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A Hierarchical Model for Random Walks in Random Media

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We show that the random walk generated by a hierarchical Laplacian in $\mathbb{Z}^d$ has standard diffusive behavior. Moreover, we show that this behavior is stable under a class of random perturbations that resemble an off-diagonal disordered lattice Laplacian. The density of states and its asymptotic behavior around zero energy are computed: singularities appear in one and two dimensions.

KEY WORDS: Random walks; random media; hierarchical model.

1. INTRODUCTION

Hierarchical models have been used in the mathematical physics literature as an intermediate step in the process of understanding the behavior of systems under renormalization group transformation. In this paper we consider the problem of a random walk in a random environment generated by a conveniently defined hierarchical Laplacian H with random coefficients, i.e., if $P_t(x, y)$ is the transition probability, then

$$\frac{\partial P_t}{\partial t}(\cdot, y) = HP_t(\cdot, y)$$

The deterministic hierarchical Laplacian is defined so as to mimic the quadratic form used by Dyson[5] in his definition of hierarchical spin systems. The coefficients of the quadratic form are chosen so that the corresponding Green's function for zero energy has the same asymptotic behavior as the usual d-dimensional lattice Laplacian as in ref. 6. This

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hierarchical Laplacian turns out to have pure point spectrum (actually, infinitely degenerate positive eigenvalues) accumulating at zero. A perhaps surprising feature of this (deterministic) operator is that, despite the nature of its spectrum, it generates a diffusion process with the same scaling properties as the usual Brownian motion:

$$P_t(Lx, Ly) = L^{-d} P_{L^{-2}t}(x, y)$$

In particular, for the mean p-displacement one obtains

$$\langle |x(t)|^p \rangle \sim C_p t^{p/2} \qquad \text{with} \quad 0 < C_p < \infty, \quad 0 < p < 2$$

The only pathology is that the mean square displacement is infinite, i.e., $C_2 = \infty$, due to the extremely nonlocal nature of the generator.

The random version of the model we consider is especially simple, as only the eigenvalues of the generator are random variables, but not their eigenfunctions as in the more realistic models discussed by Sinai[7] and Bricmont and Kupiainen.[2] In all cases considered in this paper the system exhibits standard diffusive behavior. This is to be compared with the results of Sinai,[7] who showed subdiffusive behavior in one dimension, and of Bricmont and Kupiainen,[2] who obtained diffusion in $d \geqslant 3$ for weak disorder in the asymmetric model. Our case, however, corresponds to the symmetric situation, i.e., $P_t(x, y) = P_t(y, x)$, where subdiffusive behavior has been observed only in long-range correlated environments[4] (see also ref. 2 for further references).

Our random hierarchical Laplacian corresponds to a random Schrödinger operator with off-diagonal disorder. We compute the density of states and study its asymptotic behavior as $E \to 0$ (band edge). For a class of models $\rho(E)$ is shown to be singular as $E \to 0$ in $d = 1, 2$. The associated random Schrödinger equation, however, is trivial: there are only localized states. The density of states for a hierarchical model with diagonal disorder was discussed by Bovier.[1]

This paper is organized as follows. In Section 2 we introduce the hierarchical Laplacian operator and discuss the properties of the deterministic random walk it generates. In Section 3 we consider the random diffusive process. In Section 4 we discuss the associated random Schrödinger equation and the density of states of the Hamiltonian. In the Appendix we compute the asymptotic behavior of the Green's function for $d = 1, 2$.

2. THE HIERARCHICAL LAPLACIAN

We introduce a hierarchical Laplacian on the lattice $\mathbb{Z}^d$ by defining, for $x \in \mathbb{Z}^d$, the "block" wave functions:

$$b_x^{(0)} = \delta_x \tag{2.1a}$$

$$b_x^{(1)} = L^{-d/2} \sum_{y \in B_L(x)} b_y^{(0)} \tag{2.1b}$$

and

$$\begin{aligned} b_x^{(n)} &= L^{-d/2} \sum_{y \in B_L(x)} b_y^{(n-1)} \\ &= L^{-nd/2} \sum_{y \in B_{L^n}(x)} \delta_y \end{aligned} \tag{2.1c}$$

where $B(Lx, L')$ denotes the block of center in Lx and side L' and we denote $B(Lx, L)$ by $B_L(x)$. It is convenient to take L odd and $L > 1$.

Next, we introduce the "block" operators $P_x^{(n)}$, $n = 1, 2, 3,...$, and $x \in \mathbb{Z}^d$, which project on functions in $l^2(\mathbb{Z}^d)$ with support in $B_{L^n}(x)$ and which are constant in $B(Ly, L^{n-1})$ for all $y \in B_{L^{n-1}}(x)$. In physicist's notation:

$$P_x^{(n)} = \sum_{y \in B_L(x)} |b_y^{(n-1)}\rangle\langle b_y^{(n-1)}| \tag{2.2}$$

i.e., for $\psi \in l_2(\mathbb{Z}^d)$, $n = 1, 2,...,$

$$P_x^{(n)}\psi = \sum_{y \in B_L(x)} (b_y^{(n-1)}, \psi)\, b_y^{(n-1)} \tag{2.3}$$

Finally, for $x \in \mathbb{Z}^d$, $n = 1, 2,...,$ we define the "fluctuation" orthogonal projectors

$$Q_x^{(n)} = P_x^{(n)} - |b_x^{(n)}\rangle\langle b_x^{(n)}| \tag{2.4}$$

It is important to note that $b_x^{(n)}$ and $b_y^{(n)}$, for $x \neq y$, are mutually orthogonal and dim Ran $Q_x^{(n)} = L^d - 1$. Moreover, it is easy to verify that

$$\sum_{x \in \mathbb{Z}^d} \sum_{n \geq 1} Q_x^{(n)} = \mathrm{Id} \tag{2.5}$$

i.e., $\{Q_x^{(n)}, x \in \mathbb{Z}^d, n \geq 1\}$ is a spectral partition of unity [in the sense of the strong limit of operators ion $l^2(\mathbb{Z}^d)$].

Now we can define a hierarchical Laplacian by

$$H = \sum_{x \in \mathbb{Z}^d} \sum_{n \geq 1} \alpha_x^{(n)} Q_x^{(n)} \tag{2.6}$$

where $\{\alpha_x^{(n)}, x \in \mathbb{Z}^d, n = 1, 2,...\}$ is the set of eigenvalues of H_0 and $\{Q_x^{(n)}, x \in \mathbb{Z}^d, n = 1, 2,...\}$ its spectral projections.

The first case we are going to consider is the so-called "homogeneous deterministic case" H_0, i.e., we take $\alpha_x^{(n)} = \alpha^n$, with $\alpha \in \mathbb{R}$ fixed and x-independent. The constant α will be determined by the asymptotic behavior of the Green's function at $E = 0$.

In order to justify the above-defined Laplacian, we show that it reduces to the usual hierarchical Laplacian of Dyson[5] for $d = 1$ and $L = 2$, in the sense that their quadratic forms agree: for $\varphi \in l^2(\mathbb{Z}^d)$

$$
\begin{aligned}
(\varphi, H_0 \varphi) &= \sum_{x \in \mathbb{Z}} \sum_{n \geqslant 1} \alpha^n (\varphi, Q_x^{(n)} \varphi) \\
&= \sum_{n \geqslant 1} \alpha^n \sum_{x \in \mathbb{Z}} [(\varphi, P_x^{(n)} \varphi) - (\varphi, b_x^{(n)})(b_x^{(n)}, \varphi)] \\
&= \alpha \left[\sum_{x \in \mathbb{Z}} \varphi^2(x) - L^{-d} \sum_{x \in \mathbb{Z}} \left(\sum_{y \in B_L(x)} \varphi(y) \right)^2 \right] \\
&\quad + \alpha^2 \left[L^{-d} \sum_{x \in \mathbb{Z}} \left(\sum_{y \in B_L(x)} \varphi(y) \right)^2 \right. \\
&\quad \left. - L^{-2d} \sum_{x \in \mathbb{Z}} \left(\sum_{y \in B_{L^2)x)}} \varphi(y) \right)^2 \right] + \cdots \\
&= \alpha \sum_{x \in \mathbb{Z}} \varphi^2(x) + \sum_{n \geqslant 1} \alpha^{n-1} (\alpha - 1) \sum_{x \in \mathbb{Z}} L^{-d(n-1)} \left[\sum_{y \in B_{L^{n-1}}(x)} \varphi(y) \right]^2 \\
&= \alpha \sum_{x \in \mathbb{Z}} \varphi^2(x) + \sum_{n \geqslant 1} \alpha^n (\alpha - 1) \sum_{x \in \mathbb{Z}} [\varphi_B^{(n)}(x)]^2
\end{aligned}
$$

where

$$
\varphi_B^{(n)}(x) := \sum_{y \in B_{L^n}(x)} \varphi(y)
$$

2.1. The Associated Green's Function

By definition and using the spectral theorem, for $z \in \mathbb{C} \backslash \sigma(H_0)$ and $x, y \in \mathbb{Z}^d$,

$$
\begin{aligned}
(H_0 - z)^{-1}(x, y) &= (\delta_x, (H_0 - z)^{-1} \delta_y) \\
&= \left(\delta_x, \sum_{n \geqslant 1} \sum_{\omega \in \mathbb{Z}^d} \frac{Q_\omega^{(n)}}{\alpha^n - z} \delta_y \right) \\
&= \left(\delta_x, \sum_{n \geqslant N(x, y)} \frac{Q_0^{(n)}}{\alpha^n - z} \delta_y \right) + R(x, y) \qquad (2.7)
\end{aligned}
$$

with

$$R(x, y) = \begin{cases} \displaystyle\sum_{N(x,y) \leqslant n < N(0,x)} \left(\delta_x, \frac{Q^{(n)}_{\omega_n(x,y))}}{\alpha^n - z}\delta_y\right) & \text{if} \quad N(0, x) = N(0, y) \\ 0 & \text{otherwise} \end{cases}$$

Here $N(x, y)$ denotes the smallest hierarchy which contains both x and y, i.e., the smallest block of side L^n and center in $L^n\omega$, $\omega \in \mathbb{Z}^d$, $n = 1, 2,...,$ therefore satisfying $N(x, y) \geqslant 1\ \forall x, y \in \mathbb{Z}^d$ and $N(0, 0) = 1$, and $\omega_n(x, y) \in \mathbb{Z}^d$ is uniquely determined by the conditions $x, y \in B_{L^n}(\omega_n(x, y))$ and $\omega_n(x, y) \neq 0$.

Notice that for fixed $x \in \mathbb{Z}^d$, $R(x, y) = 0$ for $|y - x|$ sufficiently large, namely, for $N(0, y) > N(0, x)$. It should also be noticed that $|x - y| \to \infty$ implies $|x - y|_h \to \infty$, but not the opposite.

It is therefore sufficient to compute the element $(0, x)$ of the associated resolvent

$$\begin{aligned}
&(H_0 - z)^{-1}(0, x) \\
&\quad = \sum_{n \geqslant N(0,x)} (\alpha^n - z)^{-1} \left\{\left[\sum_{y \in B_L(0)} (\delta_0, b_y^{(n-1)})(b_y^{(n-1)}, \delta_x)\right]\right. \\
&\qquad \left. - (\delta_0, b_0^{(n)})(b_0^{(n)}, \delta_x)\right\} \\
&\quad = \sum_{n > N(0,x)} (\alpha^n - z)^{-1}[(L^{-d/2})^{2(n-1)} - (L^{-d/2})^{2n}] \\
&\qquad - (\alpha^N - z)^{-1}(L^{-d/2})^{2N} + \delta_{0,x}(\alpha - z)^{-1} \\
&\quad = (L^d - 1) \sum_{n > N(0,x)} L^{-dn}(\alpha^n - z)^{-1} - L^{-dN}(\alpha^N - z)^{-1} + \delta_{0,x}(\alpha - z)^{-1}
\end{aligned} \tag{2.8}$$

We define the hierarchical distance $d_h(x, y)$, also denoted by $|x - y|_h$, by

$$d_h(x, y) := \begin{cases} L^{N(x,y)} & \text{for} \quad x \neq y \\ 0 & \text{if} \quad x = y \end{cases}$$

For $d > 2$, the limit $z \to 0$ gives the following result:

$$\begin{aligned}
H_0^{-1}(0, x) &= (L^d - 1) \sum_{n > N(0,x)} (\alpha L^d)^{-n} - (\alpha L^d)^{-N} + \delta_{0,x}\alpha^{-1} \\
&= \left(\frac{L^d - 1}{\alpha L^d - 1} - 1\right)(\alpha L^d)^{-N} + \delta_{0,x}\alpha^{-1} \\
&= \frac{1 - \alpha}{\alpha - L^{-d}}\alpha^{-N}(|x|_h)^{-d} + \delta_{0,x}\alpha^{-1}
\end{aligned} \tag{2.9}$$

which, for $|x| \gg 1$, is to be compared with the usual decay $|x|^{2-d}$. Therefore α must take the value L^{-2} in order that our hierarchical Laplacian has the same asymptotic behavior as the usual Laplacian at $E=0$. Notice then that $E=0$ is the only accumulation point of $\sigma(H_0)$ and all eigenvalues of H_0 are contained in the interval $(0, L^{-2})$.

Remark 1. In one and two dimensions, the Green's function must be renormalized, but we defer this to the Appendix.

Remark 2. In the case $z=-m^2<0$, the Green's function has the decay $|x|_h^{-(d+2)}$, for all d, as obtained in ref. 6 for a slightly different hierarchical Laplacian. This is to be compared with an exponential decay for the usual Laplacian.

2.2. The Associated Semigroup

As done before for the Green's function, an explicit formula for the semigroup can be readily obtained, and is given by

$$\begin{aligned} e^{-tH_0}(0, x) &:= (\delta_0, e^{-tH_0}\delta_x) \\ &= (L^d-1) \sum_{n>N(0,x)} L^{-dn} e^{-t\alpha^n} - L^{-dN} e^{-t\alpha^N} + \delta_{0,x} e^{-t\alpha}, \quad t \in \mathbb{R}^+ \end{aligned}$$

Using that

$$L^{-dN} = (L^d-1) \sum_{m>N} L^{-dm}$$

we get

$$e^{-tH_0}(0, x) = (L^d-1) \sum_{n>N(0,x)} L^{-dn}(e^{-t\alpha^n} - e^{-t\alpha^N}) + \delta_{0,x} e^{-t\alpha}$$

Now, integrating by parts, we obtain the desired expression for the semigroup,

$$\begin{aligned} e^{-tH_0}(0, x) &= \sum_{n>N(0,x)} (e^{-t\alpha^n} - e^{-t\alpha^N})(L^{-d(n-1)} - L^{-dn}) + \delta_{0,x} e^{-t\alpha} \\ &= \sum_{n \geqslant N(0,x)} L^{-dn}(e^{-t\alpha^{n+1}} - e^{-t\alpha^n}) + \delta_{0,x} e^{-t\alpha} \end{aligned} \tag{2.10}$$

The important property of this semigroup is that it defines a random walk on $\mathbb{Z}^d$, since

$$\sum_{x \in \mathbb{Z}^d} e^{-tH_0}(y, x) = 1 \qquad \forall y \in \mathbb{Z}^d, \ \forall t \in \mathbb{R}^+$$

To verify that, it is sufficient to consider the case $y=0$, the argument being as before. Therefore,

$$\sum_{x\in\mathbb{Z}^d} e^{-tH_0}(0,x)=e^{-tH_0}(0,0)+\sum_{\substack{x\in B_L(0)\\ y\neq 0}} e^{-tH_0}(0,x)+\sum_{\substack{x\in\mathbb{Z}^d\\ y\notin B_L(0)}} e^{-tH_0}(0,x)$$

$$=e^{-t\alpha}+\sum_{n\geqslant 1} L^{d(1-n)}(e^{-t\alpha^{n+1}}-e^{-t\alpha^n})$$

$$+\sum_{N\geqslant 2}(L^{dN}-L^{d(N-1)})\sum_{n\geqslant N(0,x)} L^{-dn}(e^{-t\alpha^{n+1}}-e^{-t\alpha^n})$$

Changing the order of summation in the last term of the rhs, i.e., using that

$$\frac{L^d-1}{L^d}\sum_{N\geqslant 2} L^{dN}\sum_{n\geqslant N} L^{-dn}(e^{-t\alpha^{n+1}}-e^{-t\alpha^n})$$

$$=\sum_{n=2}^{\infty} L^{-dn}(e^{-t\alpha^{n+1}}-e^{-t\alpha^n})\sum_{N=2}^{n} L^{dN}$$

we finally get that

$$\sum_{x\in\mathbb{Z}^d} e^{-tH}(0,x)=e^{-t\alpha}+L^d\sum_{n\geqslant 1} L^{-dn}(e^{-t\alpha^{n+1}}-e^{-t\alpha^n})$$

$$+\sum_{n\geqslant 2}(L^{dn}-L^d)L^{-dn}(e^{-t\alpha^{n+1}}-e^{-t\alpha^n})$$

$$=1$$

Remark 3. It can be shown in the same way that

$$\sum_{y\in\mathbb{Z}^d} f(H_0)(x,y)=1 \qquad \forall x\in\mathbb{Z}^d$$

as long as $\lim_{n\to\infty} f(\alpha^n)=1$.

2.3. Diffusive Behavior: Mean Displacement

An important quantity related to the asymptotic properties ($t\to\infty$) of the random walk is given by the mean p-displacement,

$$\sum_{x\in\mathbb{Z}^d} |x|_h^p e^{-tH_0}, \qquad t\in\mathbb{R}^+ \tag{2.10}$$

for which we can also obtain an explicit expression:

$$
\begin{aligned}
&\sum_{x\in\mathbb{Z}^d} |x|_h^p e^{-tH_0}(0,x)\\
&\quad= \sum_{\substack{x\in B_L(0)\\ x\neq 0}} L^p e^{-tH_0}(0,x) + \sum_{\substack{x\in\mathbb{Z}^d\\ x\notin B_L(0)}} L^{pN(0,x)} e^{-tH_0}(0,x)\\
&\quad= L^p(L^d-1)\sum_{n\geqslant 1} L^{-dn}(e^{-t\alpha^{n+1}}-e^{-t\alpha^n})\\
&\qquad+\sum_{N(0,x)\geqslant 2}(L^{dN(0,x)}-L^{d(N(0,x)-1)})L^{pN(0,x)}\sum_{n\geqslant N(0,x)} L^{-dn}(e^{-t\alpha^{n+1}}-e^{-t\alpha^n})\\
&\quad= L^p(L^d-1)\sum_{n\geqslant 1} L^{-dn}(e^{-t\alpha^{n+1}}-e^{-t\alpha^n})\\
&\qquad+\frac{L^d-1}{L^{d+p}-1}L^p\sum_{n\geqslant 2}(L^{(d+p)n}-L^{-dn}(e^{-t\alpha^{n+1}}-e^{-t\alpha^n})\\
&\quad= (L^d-1)\,L^{p-d}(e^{-t\alpha^2}-e^{-t\alpha})\\
&\qquad+\frac{L^{d+p}-L^p}{L^{d+p}-1}\sum_{n\geqslant 2}(L^{pn}-L^{-dn})(e^{-t\alpha^{n+1}}-e^{-t\alpha^n})
\end{aligned}\tag{2.11}
$$

where in the third equality we have changed the order of integration.

We are now in a position of presenting our first theorem.

Theorem 1. For H_0 given by (2.6) with $\alpha_x^{(n)}=\alpha^n$, $x\in\mathbb{Z}^d$, and $t\in\mathbb{R}^+$, the following holds:

$$\lim_{t\to\infty}{}^{*}\, t^{-p/2}\sum_{x\in\mathbb{Z}^d}|x|_h^p\, e^{-tH_0}(0,x)=c^*(p)<\infty,\qquad 0<p<2$$

where ($\lim^*$, $c^*(p)$) denote ($\limsup$, $\bar{c}(p)$) or (($\liminf$, $\underline{c}(p)$), with $\bar{c}(p)$ and $\underline{c}(p)$ strictly positive.

Proof. We consider $p=1$, the proof being identical for $0<p<2$. From the expression (2.11) we see that it is enough to consider the sum

$$\sum_{n\geqslant 2} L^n(e^{-t\alpha^{n+1}}-e^{-t\alpha^n})$$

all other contributions vanishing for large t. Taking $\alpha=L^{-2}$, $t=L^{2k}$, $k=1,2,\ldots$, we can readily bound the above sum from below by taking just the $(k-1)$th term, i.e.,

$$\sum_{n\geqslant 2} L^n(e^{-L^{2(k-n-1)}}-e^{-L^{2(k-n)}})\geqslant L^k\frac{1}{L}(1-e^{-L^2})$$

The upper bound is obtained by a simple application of the mean-value theorem:

$$\begin{aligned}
&\sum_{n\geqslant 2} L^n(e^{-t\alpha^{n+1}}-e^{-t\alpha^n})\\
&\quad=\left(\sum_{n=2}^{k}+\sum_{n=k+1}^{\infty}\right)[L^n(e^{-L^{2(k-n-1)}}-e^{-L^{2(k-n)}})]\\
&\quad\leqslant \sum_{n=2}^{k} L^n+\sum_{n=k+1}^{\infty}(L^{2(k-n)}-L^{2(k-n-1)})L^n\\
&\quad\leqslant\left[\frac{L}{L-1}+(1-L^{-2})\sum_{n=k+1}^{\infty}L^{k-n}\right]L^k\\
&\quad=\left(\frac{1+L-L^{-2}}{L-1}\right)L^k
\end{aligned}$$

where in the first inequality we have used that, for $n>k$,

$$|e^{-L^{2(k-n-1)}}-e^{-L^{2(k-n)}}|\leqslant|L^{2(k-n-1)}-L^{2(k-n)}|$$

Remark 4. Notice that $\forall t\in\mathbb{R}^+$, i.e., for any $k\geqslant 1$,

$$\sum_{n\geqslant 2}(e^{-L^{2(k-n-1)}}-e^{-L^{2(k-n)}}L^{2n}$$

is divergent, which in tern implies that the mean square displacement (i.e., $p=2$) is divergent: for fixed k,

$$\begin{aligned}
&\sum_{n\geqslant 2}(e^{-L^{2(k-n-1)}}-e^{-L^{2(k-n)}}L^{2n}\\
&\qquad\geqslant\sum_{n>k}(e^{-L^{2(k-n-1)}}-e^{-L^{2(k-n)}})L^{2n}\\
&\qquad\geqslant\sum_{n>k}e^{-1}(L^{2(k-n)}-L^{2(k-n-1)})L^{2n}\\
&\qquad=e^{-1}(1-L^{-2})\sum_{n>k}L^{2k}
\end{aligned}$$

where is the last inequality we have used that, for $n>k$,

$$|e^{-L^{2(k-n-1)}}-e^{-L^{2(k-n)}}|\geqslant e^{-1}|L^{2(k-n-1)}-L^{2(k-n)}|$$

Remark 5. We did not prove that $\bar{c}(p)=\underline{c}(p)$.

2.4. Scaling Properties

It is easy to show that the Green's function and the semigroup obey the following scaling relations:

$$H_0^{-1}(x, y) = L^{d-2} H_0^{-1}(Lx, Ly), \qquad x \neq y \tag{2.12a}$$

$$e^{-tH_0}(x, y) = L^{-d} e^{-L^{-2} tH_0}(L^{-1}x, L^{-1}y), \qquad x \neq y \tag{2.12b}$$

i.e., the same scaling relations as for the Green's function of the usual Laplacian.

3. THE DISORDERED CASE: THE LAPLACIAN WITH RANDOM COEFFICIENTS

3.1. Introduction

In this section we consider the hierarchical Laplacian H (2.6) with random coefficients $\alpha_x^{(n)}$ given by

$$\alpha_x^{(n)} = \alpha^n \gamma_x^{(n)} \tag{3.1}$$

with $\alpha = L^{-2}$ and $\{\gamma_x^{(n)},\ x \in \mathbb{Z}^d,\ n = 1, 2,...\}$ are independent identically distributed random variables taking values in $\mathbb{R}$ and with disorder distribution $h(\gamma)$. The reader can easily convince herself or himself that the computation of the Green's function and of the semigroup for the disordered case can be carried out in the same way as done before, obtaining that

$$\begin{aligned}\exp(-tH)(0, x) = & \sum_{n \geqslant N(0,x)} L^{-dn}[\exp(-t\alpha_0^{(n+1)}) - \exp(-t\alpha_0^{(n)})] \\ & + \delta_{0,x} \exp(-t\alpha_0^{(1)})\end{aligned}$$

and

$$\sum_{x \in \mathbb{Z}^d} e^{-tH}(y, x) = 1 \qquad \forall y \in \mathbb{Z}^d, \quad \forall t \in \mathbb{R}^+$$

where H denotes the Laplacian with random coefficients, i.e.,

$$H = \sum_{x \in \mathbb{Z}^d} \sum_{n \geqslant 1} \alpha^n \gamma_x^{(n)} Q_x^{(n)}$$

3.2. Mean 1-Displacement

Our main result is related to the asymptotic behavior ($t \to \infty$) of the mean displacement.

We consider two special cases of disorder distribution $h(\gamma)$:

(i) *Bernoulli distribution.* Here

$$\alpha_x^{(n)} = \alpha^n \gamma_x^{(n)} = \begin{cases} \alpha^n & \text{with probability} \quad p_n \\ \alpha^{n+1} & \text{with probability} \quad 1-p_n \end{cases} \tag{3.2}$$

and we take $p_n = L^{-rn}$, $r \geqslant 0$, $n = 1, 2, \ldots$. Note that $p_n \to 0$ as $n \to \infty$ and in the limiting case $p_n = 0 \; \forall n$, with probability one,

$$e^{-tH}(x, y) = L^{-2} e^{-tH_0}(x, y) \qquad \forall x, y \in \mathbb{Z}^d, \quad \forall t$$

Note also that $r = 0$ corresponds to the homogeneous deterministic case treated previously.

(ii) *Uniform distribution.* Here

$$h(\gamma_x^{(n)}) = \chi[\alpha, 1] \qquad \forall x \in \mathbb{Z}^d, \qquad n = 1, 2, \ldots \tag{3.3}$$

Remark 6. It is crucial that the values of $\{\alpha_x^{(n)},\ x \in \mathbb{Z}^d,\ n \geqslant 1\}$ are such that $P_t(0, x) \geqslant 0$ for all $x \in \mathbb{Z}^d$, $t \geqslant 0$. This is verified provided $\alpha^{n+1} \leqslant \alpha_x^{(n)} \leqslant \alpha^n$, as can be easily checked from (2.10). This explains our choices in (3.2) and (3.3).

Our main result follows:

Theorem 2. For H as given above,

$$\lim_{t \to \infty}{}^{*}\, t^{-p/2} \mathbb{E}\left\{ \sum_{x \in \mathbb{Z}^d} |x|_h^p e^{-tH}(0, x) \right\} = c^*(p) < \infty, \qquad 0 < p < 2$$

for the Bernoulli and uniform distributions, with $c^*(p) > 0$.

Remark 7. Instead of considering the Bernoulli distribution (3.2), one can also consider the renormalized Bernoulli distribution

$$\gamma^{(n)} = \begin{cases} \alpha^n / p_n & \text{with probability} \quad p_n = L^{-rn} \\ \alpha / L^{-r} & \text{with probability} \quad 1 - p_n \end{cases}$$

satisfying $\mathbb{E}\{\gamma^{(n)}\} = 1$, obtaining the same results in both cases.

Proof. First of all we compute the mean displacement for the disordered case, obtaining

$$\begin{aligned} &\sum_{x \in \mathbb{Z}^d} |x|_h^p \exp(-tH)(0, x) \\ &\quad = L^{p-d}(L^d - 1)[\exp(-t\alpha_0^{(2)}) - \exp(-t\alpha_0^{(1)})] \\ &\qquad + \frac{L^{d+p} - L^p}{L^{d+p} - 1} \sum_{n \geqslant 2} (L^{pn} - L^{-dn})[\exp(-t\alpha_0^{(n+1)}) - \exp(-t\alpha_0^{(n)})] \end{aligned}$$

Therefore, we have the following results.

(i) For the Bernoulli distribution

$$\mathbb{E}\left\{\sum_{x\in\mathbb{Z}^d}|x|_h^p e^{-tH}(0,x)\right\}$$
$$=L^{p-d}(L^d-1)[e^{-t\alpha^2}L^{-2r}$$
$$+e^{-t\alpha^3}(1-L^{-2r})-e^{-t\alpha^1}L^{-r}-(1-L^{-r})e^{-t\alpha^2}]$$
$$+\frac{L^{d+p}-L^p}{L^{d+p}-1}\sum_{n\geqslant 2}(L^{pn}-L^{-dn})[L^{-r(n+1)}e^{-t\alpha^{n+1}}$$
$$+(1-L^{-r(n+1)})e^{-t\alpha^{n+2}}-L^{-rn}e^{-t\alpha^n}-(1-L^{-rn})e^{-t\alpha^{n+1}}]$$

We can rewrite the last term of the rhs as a sum of three terms,

$$\sum_{n\geqslant 2}(L^{pn}-L^{-dn})(e^{-t\alpha^{n+2}}-e^{-t\alpha^{n+1}})$$
$$+\sum_{n\geqslant 2}(L^{pn}-L^{-dn})\,L^{-r(n+1)}(e^{-t\alpha^{n+1}}-e^{-t\alpha^{n+2}})$$
$$+\sum_{n\geqslant 2}(L^{pn}-L^{-dn})\,L^{-rn}(e^{-t\alpha^{n+1}}-e^{-t\alpha^{n}}),\qquad r>0$$

The terms with L^{-dn} make no contribution to the asymptotic limit, and the last two ones with L^{-rn} also do not contribute to the asymptotic limit, since they are of lower order than the first one:

$$\sum_{n\geqslant 2}L^{pn}(e^{-t\alpha^{n+2}}-e^{-t\alpha^{n+1}})$$

This term can be rewritten as

$$L^{-p}\sum_{n\geqslant 3}L^{pm}(e^{-t\alpha^{n+1}}-e^{-t\alpha^{n}})$$

and it is straightforward to see that it gives an upper and a lower bound of order L^{pk}, i.e., $t^{p/2}$, as in the deterministic case.

We now proceed to the second part.

(ii) For the uniform distribution, $h(\gamma_x^{(n)})=\chi_{[L^{-2},1]}$. Since

$$\mathbb{E}\{e^{-t\alpha^{(n)}}\}=\int_{L^{-2}}^{1}e^{-tL^{-2n}\gamma}\,d\gamma=\frac{L^{2n}}{t}(e^{-tL^{-2(n+1)}}-e^{-tL^{-2n}})>0$$

we get

$$\mathbb{E}\left\{\sum_{x\in\mathbb{Z}^d}|x|_h^p e^{-tH}(0,x)\right\}$$

$$= L^{p-d}(L^d-1)\left[\frac{L^4}{t}(e^{-tL^{-6}}-e^{-tL^{-4}})\right.$$

$$\left.-(e^{-tL^{-4}}-e^{-tL^{-2}})\right]+\frac{L^{d+p}-L^p}{L^{d+p}-1}\sum_{n\geq 2}(L^{pn}-L^{-dn})$$

$$\times\left[\frac{L^{2(n+1)}}{t}(e^{-tL^{-2(n+2)}}-e^{-L^{-2(n+1)}})-\frac{L^{2n}}{t}(e^{-tL^{-2(n+1)}}-e^{-tL^{-2n}})\right]$$

$$= L^{p-d+2}(L^d-1)\,L^{-2k}[L^2(e^{-tL^{-6}}-e^{-tL^{-4}})-(e^{-tL^{-4}}-e^{-tL^{-2}})]$$

$$+\frac{L^{d+p}-L^p}{L^{d+p}-1}t^{-1}\sum_{n\geq 2}(L^{(2+p)n}-L^{(2-d)n})[L^2(e^{-tL^{-2(n+2)}}-e^{-tL^{-2(n+1)}})$$

$$-(e^{-tL^{-2(n+1)}}-e^{-tL^{-2n}})]$$

As before, we can obtain a lower bound to the whole expression above by just considering the kth term of the sum, namely,

$$\frac{L^{d+p}-L^p}{L^{d+p}-1}L^{-2k}(L^{(2+p)k}-L^{(2-d)k})[L^2(e^{-L^{-4}}-e^{-L^{-2}})-(e^{-L^{-2}}-e^{-1})]$$

which in the asymptotic limit for large k gives a contribution proportional to L^{pk}, i.e., $(t^p)^{1/2}$.

To obtain an upper bound to the above expression is, in the present case, equivalent to finding an upper bound to

$$L^{-2k}\sum_{n\geq 2}L^{(2+p)n}[L^2(e^{-L^{2(k-n-2)}}-e^{-L^{2(k-n-1)}})-(e^{-L^{2(k-n-1)}}-e^{-L^{2(k-n)}})]$$

To do that we proceed as before, rewriting the sum over $n\geq 2$ as two terms and bounding them separately by

$$\sum_{n=2}^{k}L^{(2+p)n-2k}(L^2+1)+\sum_{n>k}L^{(2+p)n-2k}[L^2|L^{2(k-n-2)}-L^{2(k-n-1)}|$$

$$-\frac{1}{e}|L^{2(k-n-1)}-L^{2(k-n)}|]$$

$$= \frac{L^{2+p}}{L^{2+p}-1}(L^2+1)L^{pk} + \sum_{n>k} L^{pn}(1-L^2)(1-e^{-1})$$

$$= L^{pk}\left[\frac{L^{2+p}(L^2+1)}{L^{2+p}-1} + \frac{L^2-1}{L^2}\frac{e-1}{e}\frac{L^p}{1-L^p}\right]$$

$$= L^{pk}\left[\frac{L^{2+p}(L^2+1)}{L^{2+p}-1} - \frac{L^2-1}{L^p-1}L^{p-2}\frac{e-1}{e}\right]$$

4. SOME REMARKS ON THE SCHRÖDINGER EQUATION

4.1. Mean Square Displacement

Instead of considering the diffusion equation associated with the hierarchical Laplacian as we have until now, we will now focus on the mean square displacement of the wave function, which is given by

$$\sum_{x \in \mathbb{Z}^d} |x|_h^2 \, |e^{itH}(0, x)|^2$$

To estimate this quantity, notice that there is a trivial bound to $|e^{itH}(0, x)|$, obtained directly from the explicit representation of the semigroup (2.10),

$$\frac{2L^d}{(L^d-1)} L^{-dN(0,x)}$$

Therefore,

$$\sum_{x \in \mathbb{Z}^d} |x|_h^2 |e^{itH}(0, x)|^2$$

$$\leqslant \sum_{x \in \mathbb{Z}^d} |x|_h^2 \frac{4L^{2d}}{(L^d-1)^2} L^{-2dN(0,x)}$$

$$= L(L^d-1)\frac{4L^{2d}}{(L^d-1)^2}L^{-2d} + \sum_{N \geqslant 2} L^{(d+2)N}(1-L^{-d})L^{-2dN}\frac{4L^{2d}}{(L^d-1)^2}$$

$$= \frac{4L}{L^d-1} + \frac{4L^d}{L^d-1}\sum_{N \geqslant 2} L^{(2-d)N}$$

$$= \frac{4L}{L^d-1} + \frac{4L^{4-d}}{(L^d-1)(L^d-L^2)} \qquad (d>2) \quad \forall t$$

which is clearly finite, independent of the disorder. This means that, in a certain sense, the Schrödinger equation for the hierarchical Laplacian is trivial, since with probability one there is no dispersion of the wave function.

4.2. Density of States

As is well known, in order to compute the density of states we need the diagonal element of the integrated resolvent, given by

$$\mathbb{E}\left\{\frac{1}{\Lambda}\sum_{x\in\Lambda}(H-z)^{-1}(x,x)\right\}=(L^d-1)\sum_{n\geq 1}L^{-(d-2)n}\,\mathbb{E}\left\{\frac{1}{\gamma_0^{(n)}-z/\alpha^n}\right\}$$

after a rescaling of the energy

$$z\to z/\alpha^n$$

Therefore, a simple computation gives the density of states

$$\begin{aligned}\rho(E)&=\lim_{\varepsilon\to 0}\operatorname{Im}\mathbb{E}\{(H-z)^{-1}(0,0)\}\\&=(L^d-1)\sum_{n\geq 1}L^{-(d-2)n}h(E/\alpha^n)\end{aligned}\tag{4.1}$$

where $h(\gamma)$ denotes the disorder distribution. Let us now consider the behavior of $\rho(E)$ as $E\to 0$, The only relevant information for that is the behavior of $h(\gamma)$ at $\gamma\approx 0$. For simplicity we consider only the case $h=\chi_{[0,1]}$. Taking $E_m=L^{-2m}$ we now compute

$$\rho(L^{-2m})=(L^d-1)\sum_{n=1}^{m}L^{-(d-2)n}$$

and therefore we get for

(i) $d=1$

$$\lim_{m\to\infty}L^{-m}\rho(L^{-2m})=\frac{-L+1}{L^{-1}-1}=L$$

i.e., $\lim_{E\to 0}\sqrt{E}\,\rho(E)=L$,

(ii) For $d=2$

$$\begin{aligned}&\lim_{m\to\infty}m^{-1}\rho(L^{-2m})\\&\quad=\lim_{m\to\infty}m^{-1}(L^d-1)L^{-(d-2)m}[L^{(d-2)(m-1)}+L^{(d-2)(m-2)}+\cdots+1]\\&\quad=\lim_{m\to\infty}m^{-1}(m-1)(L^2-1)=L^2-1\end{aligned}$$

i.e.,

$$\lim_{E\to 0}(\log_L E)^{-1}\,\rho(E)=L^2-1$$

(iii) For $d=3$

$$
\begin{aligned}
L^m \rho(L^{-2m}) &= L^m (L^3 - 1) \sum_{n=1}^{m} L^{-n} \\
&= \frac{L^3 - 1}{L - 1} L^m (1 - L^{-m}) \\
&= -(L^2 + L + 1) + (L^2 + L + 1) L^m
\end{aligned}
$$

i.e.,

$$\rho(L^{-2m}) \underset{m \to \infty}{\sim} (L^2 + L + 1)$$

Remarks 8. Notice that the density of states of the deterministic problem is obtained from (4.1) by the replacement of $h(E/\alpha^n)$ by $\delta(E/\alpha^n)$. We therefore see that the introduction of disorder produces a smoothing out of the asymptotic behavior around $E=0$. The remaining singular behavior in $d=1, 2$ as $E \to 0$ is the same as for the deterministic Laplacian in $\mathbb{R}^d$, and it appears as long as the origin $E=0$ belongs to the support of h.

Remark 9. The usual random Schrödinger operator with off-diagonal disorder in $d=1$ has been shown to have a singularity in the density of states at the band center[3] which has no apparent connection to the singularities observed here.

Remark 10. It is interesting to compare the above results with the behavior of the density of states for the model with diagonal disorder, $H_0 + V$, where $\{V(x),\ x \in \mathbb{Z}^d\}$ are independent identically distributed random variables with common distribution $d\mu(V)$ and with H_0 as before. In the latter case if $d\mu(V)$ is absolutely continuous, $d\mu(V) = f(V)\,dV$, with bounded derivative $f(V)$, then we can use a theorem by Wegner[8] to obtain that the density of states $\rho(E)$ is also a bounded function.

APPENDIX

As with the usual Laplacian, in one and two dimensions it is necessary to renormalize the Green's function so as to avoid divergences in the thermodynamic limit. This is done by subtracting the diagonal part of the resolvent, i.e.,

$$H_R^{-1}(0, x) = \lim_{\Lambda \to \infty} H_{R,\Lambda}^{-1}(0, x) = \lim_{\Lambda \to \infty} [H_{0,\Lambda}^{-1}(0, x) - H_{0,\Lambda}^{-1}(0, 0)]$$

Recall that in a finite volume $\Lambda = L^M$ and in the limit $z \to 0$

$$H_{0,\Lambda}^{-1}(0, 0) = (L^d - 1) \sum_{n=1}^{M} L^{(2-d)n}$$

and therefore, for $x \neq 0$,

$$\begin{aligned} H_{R,\Lambda}^{-1}(0, x) &= (L^d - 1) \sum_{n=N(0,x)+1}^{M} L^{(2-d)n} - L^{(2-d)N(0,x)} \\ &\quad - (L^d - 1) \sum_{n=1}^{M} L^{(2-d)n} \\ &= (L^d - 1) \sum_{n=1}^{N(0,x)} L^{(2-d)n} - L^{(2-d)N(0,x)} \end{aligned}$$

We then obtain the following asymptotic behavior as $|x| \to \infty$ in the thermodynamic limit:

(i) For $d = 1$

$$\begin{aligned} H_R^{-1}(0, x) &= \lim_{\Lambda \to \mathbb{Z}} H_{R,\Lambda}^{-1}(0, x) \\ &= (L - 1) \sum_{n=1}^{N(0,x)} L^n - L^{N(0,x)} \\ &= \frac{L-1}{1-L} (L - L^{N(0,x)+1}) - L^{N(0,x)} \\ &= L^{N(0,x)}(L-1) - L \underset{|x|_h \to \infty}{\sim} (L-1)|x|_h \end{aligned}$$

(ii) For $d = 2$

$$\begin{aligned} H_R^{-1}(0, x) &= (L^2 - 1) \sum_{n=1}^{N(0,x)} 1 - 1 \\ &= (L^2 - 1) N(0, x) - 1 \underset{m \to \infty}{\sim} (L^2 - 1) \log_L |x|_h \end{aligned}$$

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